Instrumentation Physics for Eddy Covariance

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November 2013

Instrument characteristics

Performance characteristics are of two types: static and dynamic.

The static characteristics of an instrument or measurement system provide a quantitative measures of how well that instrument measures its primary inputs. The dynamic characteristics determine how that instrument responds to spurious, erratic, or impulsive input. In the simplest terms, an instrument's static characteristics are usually summarized by or are synonymous with the calibration curve. Whereas, the dynamic characteristics is usually described by a response function or a transfer function.

Determining an instrument's calibration curve is usually straightforward and normally involves linear regression of the measured output against a known input. The dynamic response of an instrument is often more difficult to determine and typically involves a mathematical model derived from the physical principles underlying the instrument's measurement technology. In turn this yields a mathematical analog of the instrument's response to the dynamic input (or forcing function).

This notes focus on the dynamic response of eddy covariance instrumentation.

A few definitions:

In the simplest cases the basic physical principles underlying an instrument's measurement technology can be described with an linear ordinary differential equation (LODE). In general, therefore, the dynamic response of an instrument can be described as:

$$a_n \frac{d^n X_O}{dt^n} + a_{n-1} \frac{d^{n-1} X_O}{dt^{n-1}} + \cdots + a_1 \frac{dX_O}{dt} + X_O = X_I(t)$$

where t = time, $X_O = X_O(t)$ = the output signal of the instrument, $X_I(t)$ = the input signal or the forcing function, and a_n, a_{n-1}, ..., a_1 are constants that characterize the physical properties of the instrument.

This equation is a LODE of order n. Therefore, we can say that this LODE describes an n^{th}-order instrument.

Again in general terms, an instrument's transfer function or response function or filter function (h) is defined as:

$$h(a_n, a_{n-1}, \ldots, a_0;\, t) = X_O(t)/X_I(t)$$

Examples of first and second order instruments

Good examples of first and second order systems or instruments can be found in some simple mechanical (meteorological) devices and elementary electrical circuits.

For example, the equation of motion for a cup anemometer is: $\mathcal{I} dv/dt - fv = F(t)$; where $\mathcal{I}$ is moment of inertia of the cup anemometer, v is rotational velocity, f is a friction factor associated with the bearings, and $F(t)$ is the external force (torque) of the wind acting on the cup, which for the general case should be considered a function of time. In this simple case $\mathcal{I} dv/dt$ is the rotational acceleration of the cup and fv is the frictional drag force. The cup anemometer is, therefore, a first order instrument in velocity.

However, in general mechanical devices are second order systems. Consider the wind vane, which is a special case of the harmonic oscillator. The equation of motion for a harmonic oscillator is second order in displacement: $m\, d^2x/dt^2 - f dx/dt + kx = F(t)$; where x is displacement. Here the new term kx is the restoring force associated with displacement from equilibrium. In terms of energy of the system: the acceleration term is related to the change in kinetic energy, the drag force is related to the rate of energy dissipation, the displacement force is related to the potential energy of the system, and finally the forcing term is related to the external energy being input into the system.

First order systems: LR and RC alternating current circuits

The LR alternating electrical current circuit is described by

$$L\frac{dI}{dt} + R\,I = V_0 e^{-i\omega t}$$

while the RC alternating current circuit is described by

$$R\frac{dI}{dt} + \frac{I}{C} = -i\omega V_0 e^{-i\omega t}$$

where I is the current through the circuit, R is the ohmic resistance, C is the electrical capacitance, L is the electrical inductance, and V_0 is the amplitude of the applied voltage, which has a driving frequency ω. Here i is the unit imaginary number $\sqrt{-1}$.

Note: I am assuming some familiarity with complex numbers because complex variables are very economical for determining the amplitude and phase response of an instrument.

The steady state solutions to these equations (i.e., $I(t) = I_0 e^{-i\omega t}$) produces the following response functions:

$$h_{LR}(\omega) = \frac{1}{1 - i\omega\tau}$$

for the LR circuit and

$$h_{RC}(\omega) = \frac{1}{1 + 1/(-i\omega\tau)} = \frac{-i\omega\tau}{1 - i\omega\tau}$$

for the RC circuit. Where τ is the time constant for each circuit. For the LR circuit $\tau = L/R$ and for the RC circuit $\tau = RC$.

Although both systems are first order systems there response characteristics are rather different. This is seen by comparing the gain function $= H(\omega) = |h(\omega)| = \sqrt{h(\omega)h^*(\omega)}$ of each system. Where $h^*(\omega)$ is the complex conjugate of $h(\omega)$. In the case of the LR circuit

$$H_{LR}(\omega) = \frac{1}{\sqrt{1 + \omega^2\tau^2}}$$

whereas for the RC circuit

$$H_{RC}(\omega) = \frac{\omega\tau}{\sqrt{1 + \omega^2\tau^2}}$$

It is clear from these expressions that the LR circuit attenuates high frequencies ($\omega > 1/\tau$), whereas the RC circuit attenuates low frequencies ($\omega < 1/\tau$). So we call the LR circuit a low-pass filter and the RC circuit a high-pass filter. Furthermore, we term the frequency at which $H^2(\omega) = 1/2$ the half-power point and denote it by $\omega_{1/2}$. For these simple filters, $\omega_{1/2} = 1/\tau$.

The phase response, $\phi(\omega)$, of these two filters is determined by expressing $h(\omega)$ as:

$$h(\omega) = H(\omega)e^{i\phi(\omega)}$$

Therefore,

$$\phi_{LR}(\omega) = \tan^{-1}(\omega\tau)$$

and

$$\phi_{RC}(\omega) = -\tan^{-1}(1/\omega\tau)$$

Transients and impulsive responses

To complete the description of the first order low pass instrument some discussion of the transient response and the response to impulsive forcing is necessary. In the most general terms, LODE that describes this type of instrument is

$$\tau \frac{dX_O}{dt} + X_O = X_I(t)$$

Assuming that the initial state (defined for all $t < 0$) is $X_I = X_O = X_{in}$ and that at time $t = 0$ the forcing is "switched on". In the case of sinusoidal forcing assume that $X_I(t) = X_{in} + A_I e^{-i\omega t}$, where A_I is the amplitude of the forcing. The complete solution to this equation with these initial conditions is:

$$X_O(t) = X_{in} + \frac{A_I}{1 - i\omega\tau} e^{-i\omega t} - \frac{A_I}{1 - i\omega\tau} e^{-t/\tau}$$

where the second term of the right is the steady state solution and the third term on the right is the transient. In this case the transfer function is $h(\omega, \tau; t) = [X_O(t) - X_{in}]/[X_I(t) - X_{in}]$, or

$$h(\omega, \tau; t) = \frac{1 - e^{-t/\tau} e^{i\omega t}}{1 - i\omega\tau}$$

In the case of an impulsive response (or step function input with amplitude A_I), then the response of a first order low-pass system or instrument is described by the following two expressions

$$X_O(t) = X_{in} + A_I - A_I e^{-t/\tau}$$

and

$$h(\tau; t) = 1 - e^{-t/\tau}$$