

## Logistic Growth

Consider the growth of (a population of) plants,  $P$ . If the probability of reproduction per unit time is  $g$ , then the population changes according to

$$\frac{dP}{dt} = gP . \quad (1)$$

Equation (1) describes exponential growth. Taking the integral of both sides, the solution is

$$P(t) = P(0)e^{gt} \quad (2)$$

where  $P(0)$  is the initial population (at time=0). According to (2),  $P$  will grow very quickly at an accelerating rate, without bound.

In the real world nothing can grow exponentially forever without “running out” of something (space, light, water, nutrients, etc). A simple way to account for limits to growth, we can modify the growth rate so that it decreases as the population approaches some limit imposed by the environment, for example by a limiting nutrient:

$$\frac{dP}{dt} = g\left(1 - \frac{P}{K}\right)P . \quad (3)$$

It's helpful to think of  $g\left(1 - \frac{P}{K}\right)$  as the modified or constrained growth rate, which is the probability of reproduction for each individual in the population per time step. When  $P$  is small enough so that  $P/K \ll 1$ , the solution to (3) is very close to (2) and the population grows exponentially. As  $P$  gets closer to  $K$ , the population grows more slowly, and in the limit as  $P \rightarrow K$  the modified growth rate approaches zero.

Equation (3) is known as the logistic growth equation. It's very commonly used in biology and ecology to represent population growth. You can read more about it all over the web, but a good place to start is [Wikipedia](#). In the traditional formulation of logistic growth,  $K$  is called the “*carrying capacity*” of the system. In reality the carrying capacity is some kind of complicated function of the physical and biological conditions in which the population develops, but for our purposes we can just think of it as a number.

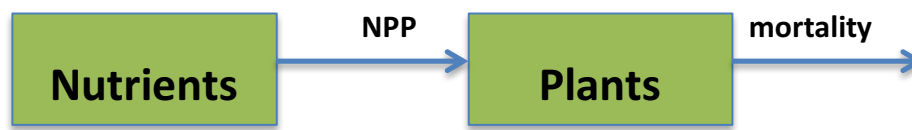
It might seem a little depressing, but let's add death to this problem. Think of the rate of death as being equal to the intrinsic (unconstrained) rate of growth divided by the average lifetime  $L$ , so that in each time step the probability of an individual dying is  $g/L$ .

So we can think of the overall growth model as

$$\frac{dP}{dt} = g\left(1 - \frac{P}{K} - \frac{1}{L}\right)P. \quad (4)$$

In (4), let  $P$  = plant carbon;  $g$  = the intrinsic growth rate of plant carbon;  $K$  = resource-limited max of plant carbon ("Carrying capacity"); and  $L$  = relative "lifetime" of plant carbon (so that the death rate  $d = g/L$ ).

You can think of equation (4) as a global model of the growth and death of living plant carbon or biomass. It looks something like this:



If we start with a nonzero stock of plant carbon and let equation (4) run for a long time, it will approach a steady state at which time  $dP/dt = 0$ .

So here's what I want you to do:

1. Set equation (4) to zero and solve for the steady state carbon pool  $P$  as a function of the parameters  $K$  and  $L$ .
2. In the real world, we estimate the global stock of plant carbon to be about 500 GtC, and the global annual Net Primary Production (NPP) to be 60 GtC/yr. Assuming the average lifetime of a plant is 3 years, find values for the carrying capacity  $K$  and intrinsic growth rate  $g$  so that the steady state NPP and global plant carbon of your model agree with these "observed" values. Show your work.
3. Make a simple computer program out of your version of equation (4) with appropriate parameter values. Test it with initial carbon pools both lower and higher than 500 GtC, and ensure that it does in fact converge on the correct solution after a long time. Hand in a printout of your code and a couple of plots to demonstrate that it's working.